Definition of Inverse Function

A function and its inverse function can be described as the "DO" and the "UNDO" functions. A function takes a starting value, performs some operation on this value, and creates an output answer. The inverse function takes the output answer, performs some operation on it, and arrives back at the original function's starting value.

This "DO" and "UNDO" process can be stated as a composition of functions. If functions *f* and *g* are inverse functions, $f(g(x)) = g(f(x)) = x$. A function composed with its inverse function yields the original starting value. Think of them as "undoing" one another and leaving you right where you started.

So how do we find the inverse of a function?

Basically speaking, the process of finding an inverse is simply the swapping of the *x* and *y* coordinates. This newly formed inverse will be a **relation**, but may **not** necessarily be a function.

Consider this subtle difference in terminology:
Definition: INVERSE OF A FUNCTION: The relation formed when the independent variable is exchanged with the dependent variable in a given relation. *(This inverse may NOT be a function.)*

Definition: INVERSE FUNCTION: If the above mentioned inverse of a function is itself a function, it is then called an *inverse function.*

Remember:

The inverse of a function may not always be a *function*!

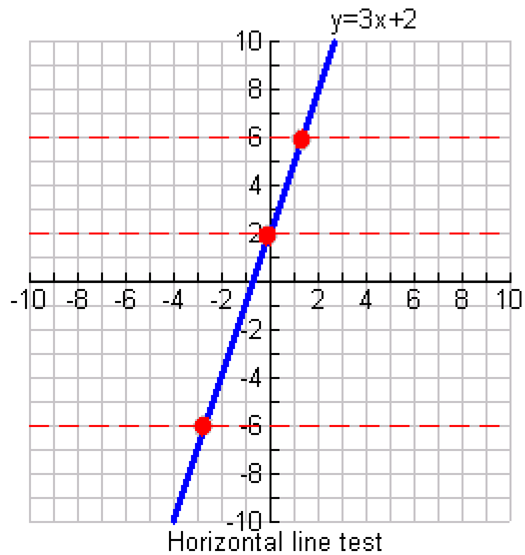
The original function must be a *one-to-one function* to guarantee that its inverse will also be a function.

Definition: A function is a ***one-to-one function*** if and only if each second element corresponds to one and only one first element. (each x and y value is used only once)

Use the **horizontal line test** to determine if a function is a *one-to-one function*. If ANY horizontal line intersects your original function in ONLY ONE location, your function will be a one-to-one function and its inverse will also be a *function*.

The function $y = 3x + 2$, shown at the right, IS a one-to-one function and its inverse will also be a *function*.

(Remember that the **vertical line test** is used to show that a relation is a function.)



Horizontal line test

Definition: The *inverse* of a function is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original function.

Should the inverse of function $f(x)$ also be a function, this inverse function is denoted by $f^{-1}(x)$.

Note: If the original function is a *one-to-one function*, the inverse will be a *function*.

[The notation $f^{-1}(x)$ refers to "inverse function". It does not algebraically mean $1/f(x)$.]

If a function is composed with its inverse function, the result is *the starting value*. Think of it as the function and the inverse undoing one another when composed. Consider the simple function $f(x) = \{(1,2), (3,4), (5,6)\}$ and its inverse $f^{-1}(x) = \{(2,1), (4,3), (6,5)\}$

$$(f \circ f^{-1})(x) = \{(2,2),(4,4),(6,6)\}$$

More specifically:

$$(f \circ f^{-1})(2) = f(f^{-1}(2))$$
$$= f(1) = 2$$

The answer is the starting value of 2.

"So, how do we find inverses?"

Consider the following three solution methods:

Swap ordered pairs: If your function is defined as a list of ordered pairs, simply swap the *x* and *y*values. Remember, the inverse relation will be a *function* only if the original function is one-to-one.

Examples:

a. Given function *f*, find the inverse relation. Is the inverse relation also a *function*?

$f(x) = \{(3,4),(1,-2),(5,-1),(0,2)\}$

Answer:
Function *f* is a one-to-one function since the *x* and *y* values are used only once. Since function *f* is a one-to-one function, the inverse relation is also a function.
Therefore, the inverse function is:

$f^{-1}(x) = \{(4,3),(-2,1),(-1,5),(2,0)\}$

b. Determine the inverse of this function. Is the inverse also a *function*?

x	1	-2	-1	0	2	3	4	-3
f (*x*)	2	0	3	-1	1	-2	5	1

Answer: Swap the *x* and *y* variables to create the inverse relation. The inverse relation will be the set of ordered pairs:
{(2,1), (0,-2), (3,-1), (-1,0), **(1,2)**, (-2,3), (5,4),**(1,-3)**}
Since function *f* was **not** a one-to-one function (the *y* value of 1 was used twice), the inverse relation will **NOT** be a function (because the *x* value of 1 now gets mapped to two separate *y* values which is not possible for functions).

Solve algebraically: Solving for an inverse relation algebraically is a three step process:

1. Set the function = y
2. Swap the x and y variables
3. Solve for y

Examples:

a. Find the inverse of the function $f(x) = x - 4$

Answer:

$y = x - 4$	*Remember:* Set = *y*.
$x = y - 4$	Swap the variables.
$x + 4 = y$	Solve for *y*.
$f^{-1}(x) = x + 4$	Use the inverse function notation since *f* (*x*) is a one-to-one function.

b. Find the inverse of the function $f(x) = \dfrac{x+1}{x}$ (given that *x* is not equal to 0).

Answer:

$y = \dfrac{x+1}{x}$	*Remember:* Set = *y*.
$x = \dfrac{y+1}{y}$	Swap the variables.
$xy = y + 1$	Eliminate the fraction by multiplying each side by *y*.
$xy - y = 1$	Get the *y*'s on one side of the equal sign by subtracting *y* from each side.
$y(x-1) = 1$	Isolate the *y* by factoring out the *y*.
$y = \dfrac{1}{x-1}$	Solve for *y*.
$f^{-1}(x) = \dfrac{1}{x-1}$	Use the inverse function notation since *f* (*x*) is a one-to-one function.

Graph: The graph of an inverse relation is the reflection of the original graph over the identity line,
y = *x*. It may be necessary to restrict the domain on certain *functions* to guarantee that the inverse relation is also a *function*. (Read more about graphing inverses.)

Example:

Consider the straight line, ***y* = 2*x* + 3**, as the original function. It is drawn in **blue**.

If reflected over the identity line, *y* = *x*, the original function becomes the **red** dotted graph. The new **red** graph is also a straight line and passes the vertical line test for functions. The inverse relation of *y* = 2*x* + 3 is also a function.

Not all graphs produce an inverse relation which is also a*function*.