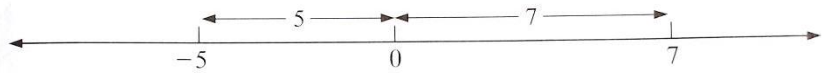
The **modulus** of a real number x is its distance from 0 on the number li

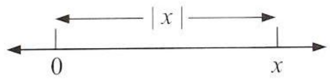
Because the modulus is a distance, it cannot be negative.



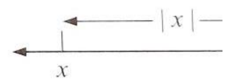
So, the modulus of 7 is 7, which is written as $|7| = 7$
and the modulus of -5 is 5, which is written as $|-5| = 5$.

Thus, $|x|$ is the distance of x from 0 on the number line.

If $x > 0$



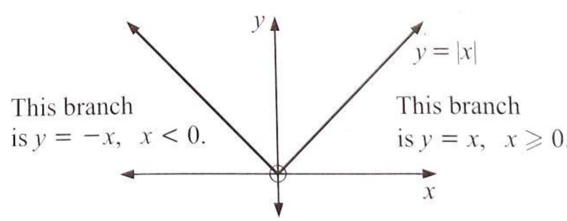
If $x < 0$



ALGEBRAIC DEFINITION

The modulus of x, $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$

$y = |x|$ has graph



Notice that $\sqrt{7^2} = \sqrt{49} = 7$ and $\sqrt{(-5)^2} = \sqrt{25} = 5$

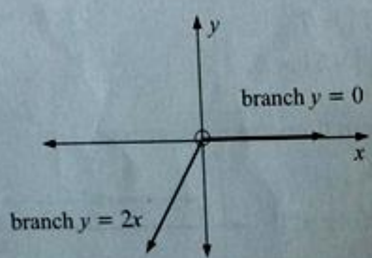
Thus $|x| = \sqrt{x^2}$ is an equivalent definition of $|x|$.

By replacing $|x|$ with x for $x \geqslant 0$ and $-x$ for $x < 0$, write the following functions without the modulus sign and hence graph each function:

a $f(x) = x - |x|$ **b** $f(x) = x|x|$

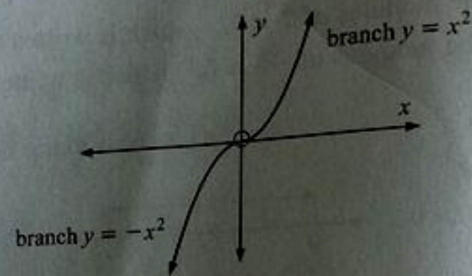
a If $x < 0$, $f(x) = x - (-x) = 2x$.
If $x \geqslant 0$, $f(x) = x - x = 0$.

So, we graph $\begin{cases} y = 2x & \text{for } x < 0 \\ y = 0 & \text{for } x \geqslant 0. \end{cases}$



b If $x \geqslant 0$, $f(x) = x(x) = x^2$.
If $x < 0$, $f(x) = x(-x) = -x^2$.

So, we graph $\begin{cases} y = x^2 & \text{for } x \geqslant 0 \\ y = -x^2 & \text{for } x < 0. \end{cases}$



EXERCISE 1F.1

1 If $a = -2$, $b = 3$, $c = -4$ find the value of:

a $|a|$ **b** $|b|$ **c** $|a||b|$ **d** $|ab|$
e $|a-b|$ **f** $|a|-|b|$ **g** $|a+b|$ **h** $|a|+|b|$
i $|a|^2$ **j** a^2 **k** $\left|\dfrac{c}{a}\right|$ **l** $\dfrac{|c|}{|a|}$

2 If $x = -3$, find the value of:

a $|5-x|$ **b** $|5|-|x|$ **c** $\left|\dfrac{2x+1}{1-x}\right|$ **d** $|3-2x-x^2|$

3 **a** Is $|a+b| = |a|+|b|$? **b** Is $|a-b| = |a|-|b|$?

4 Copy and complete:

a	b	$\lvert ab \rvert$	$\lvert a \rvert \lvert b \rvert$	$\left\lvert \frac{a}{b} \right\rvert$	$\frac{\lvert a \rvert}{\lvert b \rvert}$
6	2				
6	-2				
-6	2				
-6	-2				

What do you suspect?

5 Use the fact that $\lvert x \rvert = \sqrt{x^2}$ to prove that:

a $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$ **b** $\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$, $b \neq 0$ **c** $\lvert a-b \rvert = \lvert b-a \rvert$

6 Using $\lvert a \rvert = a$ if $a \geqslant 0$ and $-a$ if $a < 0$, write the following functions without modulus signs and hence graph each function:

a $y = \lvert x-2 \rvert$ **b** $y = \lvert x+1 \rvert$ **c** $y = -\lvert x \rvert$

d $y = \lvert x \rvert + x$ **e** $y = \frac{\lvert x \rvert}{x}$ **f** $y = x - 2\lvert x \rvert$

g $y = \lvert x \rvert + \lvert x-2 \rvert$ **h** $y = \lvert x \rvert - \lvert x-1 \rvert$ **i** $y = \lvert x^2+1 \rvert$

j $y = \lvert x^2-1 \rvert$ **k** $y = \lvert x^2-2x \rvert$ **l** $y = \lvert x^2+3x+2 \rvert$

MODULUS EQUATIONS

From the previous exercise you should have discovered these **properties of modulus**:

- $\lvert x \rvert \geqslant 0$ for all x
- $\lvert -x \rvert = \lvert x \rvert$ for all x
- $\lvert x \rvert^2 = x^2$ for all x
- $\lvert xy \rvert = \lvert x \rvert \lvert y \rvert$ for all x and y
- $\left\lvert \frac{x}{y} \right\rvert = \frac{\lvert x \rvert}{\lvert y \rvert}$ for all x and y, $y \neq 0$
- $\lvert a-b \rvert = \lvert b-a \rvert$ for all a and b.

It is clear that $\lvert x \rvert = 2$ has two solutions, $x = 2$ and $x = -2$.

In general, if $\lvert x \rvert = a$ where $a > 0$, then $x = \pm a$.

We use this rule to solve modulus equations.

Example 15

Solve for x: **a** $\lvert 2x+3 \rvert = 7$ **b** $\lvert 3-2x \rvert = -1$

a $\lvert 2x+3 \rvert = 7$

$\therefore\ 2x+3 = \pm 7$

$\therefore\ 2x = 7-3$ or $-7-3$

$\therefore\ 2x = 4$ or -10

$\therefore\ x = 2$ or -5

b $\lvert 3-2x \rvert = -1$

has no solution as LHS is never negative.

Solve for x: $\left|\dfrac{3x+2}{1-x}\right| = 4$

$$\left|\frac{3x+2}{1-x}\right| = 4 \quad \therefore \quad \frac{3x+2}{1-x} = \pm 4$$

If $\dfrac{3x+2}{1-x} = 4$

then $3x+2 = 4(1-x)$

$\therefore \; 3x+2 = 4-4x$

$\therefore \; 7x = 2$

$\therefore \; x = \frac{2}{7}$

If $\dfrac{3x+2}{1-x} = -4$

then $3x+2 = -4(1-x)$

$\therefore \; 3x+2 = -4+4x$

$\therefore \; 6 = x$

So, $x = \frac{2}{7}$ or 6.

Also notice that if $|x| = |b|$ then $x = \pm b$.

Example 17

Solve for x: $|x+1| = |2x-3|$

If $|x+1| = |2x-3|$, then $x+1 = \pm(2x-3)$ {using property above}

If $x+1 = 2x-3$

then $4 = x$

If $x+1 = -(2x-3)$

then $x+1 = -2x+3$

$\therefore \; 3x = 2$

$\therefore \; x = \frac{2}{3}$

So, $x = \frac{2}{3}$ or 4.

EXERCISE 1F.2

1 Solve for x:

a $|x| = 3$
b $|x| = -5$
c $|x| = 0$
d $|x-1| = 3$
e $|3-x| = 4$
f $|x+5| = -1$
g $|3x-2| = 1$
h $|3-2x| = 3$
i $|2-5x| = 12$

2 Solve for x:

a $\left|\dfrac{x}{x-1}\right| = 3$
b $\left|\dfrac{2x-1}{x+1}\right| = 5$
c $\left|\dfrac{x+3}{1-3x}\right| = \dfrac{1}{2}$

3 Solve for x:

a $|x+1| = |2-x|$
b $|x| = |5-x|$
c $|3x-1| = |x+2|$
d $|2x+5| = |1-x|$
e $|1-4x| = 2|x-1|$
f $|3x+2| = 2|2-x|$

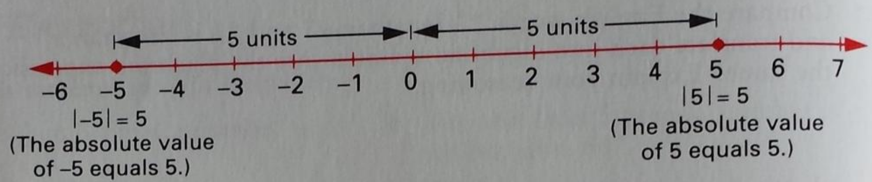
4 Solve for x using **i** a graphical method **ii** an algebraic method:

a $|x+2| = 2x+1$
b $|2x+3| = 3|x|-1$
c $|x-2| = \frac{2}{5}x+1$

5.6 Equations with Absolute Value

Objective To solve equations containing absolute value

Recall (Lesson 2.2) that the absolute value of a real number is the distance between the number and 0 on a number line. Thus, 5 and -5 both have an absolute value of 5, since both are 5 units from 0.



The equation $|x| = 5$ can be written as the disjunction $x = 5$ *or* $x = -5$. It has two solutions, 5 and -5. Since an absolute value cannot be negative, an equation such as $|x| = -3$ has no solution.

EXAMPLE 1 Solve each equation: **a.** $|x - 1| = 3$ **b.** $|9 - 2x| = 15$

Plan Write the equation as a disjunction and solve.

Solutions

a. $x - 1 = 3$ *or* $x - 1 = -3$
$x = 4$ $\quad$ $x = -2$

b. $9 - 2x = 15$ *or* $9 - 2x = -15$
$-2x = 6$ $\quad$ $-2x = -24$
$x = -3$ $\quad$ $x = 12$

Checks

a. $|x - 1| = 3$

$|x - 1| = 3$ $\qquad$ $|x - 1| = 3$
$|4 - 1| \stackrel{?}{=} 3$ $\qquad$ $|-2 - 1| \stackrel{?}{=} 3$
$|3| \stackrel{?}{=} 3$ $\qquad$ $|-3| \stackrel{?}{=} 3$
$3 = 3$ $\qquad$ $3 = 3$

Thus, the solutions are -2 and 4.

b. $|9 - 2x| = 15$

$|9 - 2(-3)| \stackrel{?}{=} 15$ $\qquad$ $|9 - 2x| = 15$
$|9 + 6| \stackrel{?}{=} 15$ $\qquad$ $|9 - 2(12)| \stackrel{?}{=} 15$
$|15| \stackrel{?}{=} 15$ $\qquad$ $|9 - 24| \stackrel{?}{=} 15$
$15 = 15$ $\qquad$ $|-15| \stackrel{?}{=} 15$
$15 = 15$

Thus, the solutions are -3 and 12.

TRY THIS Solve each equation. **1.** $|x + 5| = 8$

2. $|4x - 6| = 46$

EXAMPLE 2 Solve $-2|3y + 5| + 4 = 2$.

Solution

Subtract 4 from each side. $-2|3y + 5| + 4 = 2$

Divide each side by -2. $-2|3y + 5| = -2$

$|3y + 5| = 1$

Write a disjunction. $3y + 5 = 1$ *or* $3y + 5 = -1$

$3y = -4$ $3y = -6$

$y = -\frac{4}{3}$ $y = -2$

Thus, the solutions are $-\frac{4}{3}$ and -2. The check is left for you.

TRY THIS 3. Solve $8 + 5|6 - 2y| = 68$.

Classroom Exercises

Solve each equation.

1. $|x| = 6$
2. $|y| = -1$
3. $|z| = 0$
4. $|d| + 1 = 4$
5. $2|b| = -8$
6. $|y - 6| = 10$
7. $|2y| = 10$
8. $|4x - 1| = 5$

Written Exercises

Solve each equation.

1. $|x| = 4$
2. $|x| = -4$
3. $2|x| = 6$
4. $-3|x| = -12$
5. $|x - 4| = 5$
6. $|y - 3| = 7$
7. $|y + 8| = -3$
8. $|5 - z| = 3$
9. $|-5c| = 15$
10. $5|-c| = 15$
11. $|3y + 1| = -2$
12. $2|d + 2| = 14$
13. $5|x - 2| - 13 = 7$
14. $4|y - 3| - 2 = 10$
15. $3|z + 5| + 5 = 8$
16. $-6 - 3|4x - 2| = 8$
17. $10 = 6|a - 7| - 14$
18. $1 = 8 - |2y - 5|$

Complete each sentence. Use $>$, $<$, or $=$.

19 The solution set of $|x| = a$ is $\{0\}$ if a __?__ 0.

20. The solution set of $|x| = a$ is $\varnothing$ if a __?__ 0.

Mixed Review

Solve. 3.2, 3.5, 4.6

1. $\frac{1}{2}x = \frac{1}{3}$ 2. $4(x + 2) = 3x - 9$ 3. $3x + 1 = -(4 - x)$ 4. $-0.2 = y + 4.9$

5. Last year a particular car cost $10,000. This year the same model costs $10,600. What is the percent increase in the cost? 4.7